AACPP - Session 8

Assignment 3 Solution, Assignment 4 Intro

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Assignment 4 - DMC

Assignment 4 Rewind

Input

5 6
5 3
2
4
8
3
10
1 5
1 2
2 4
4 5
2 3
3 4

Output

5
1 4

Assignment 4 Rewind

Input

5 6
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2
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8
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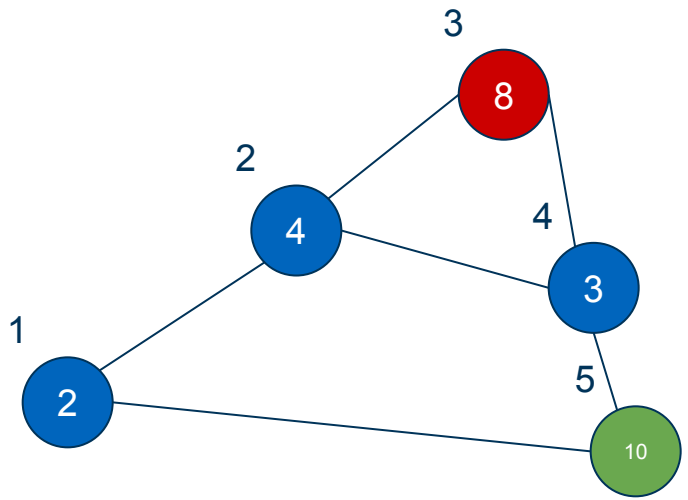

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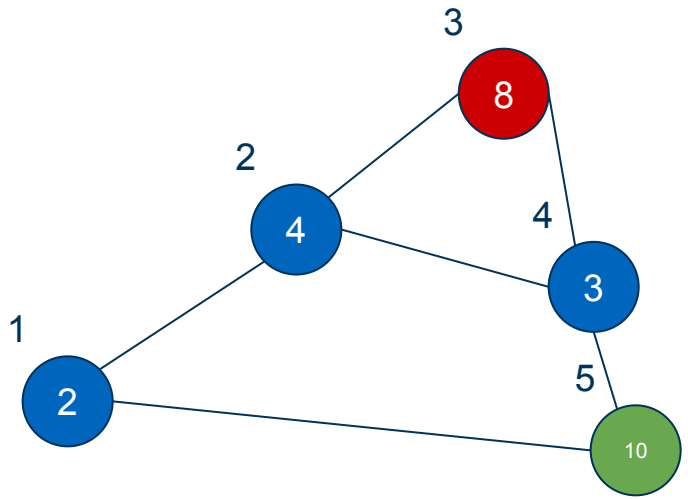

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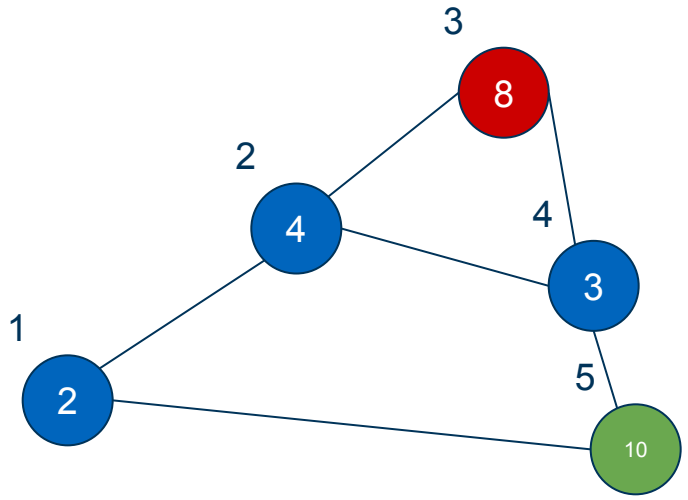

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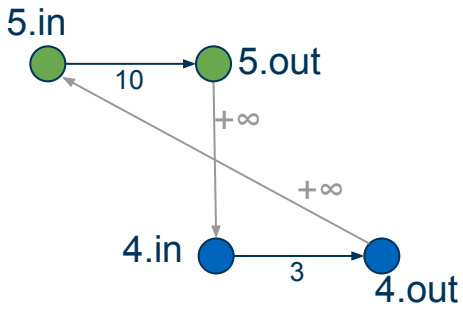

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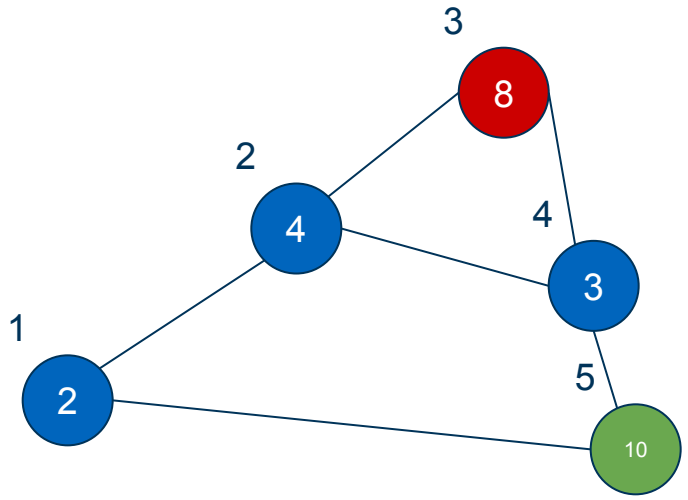

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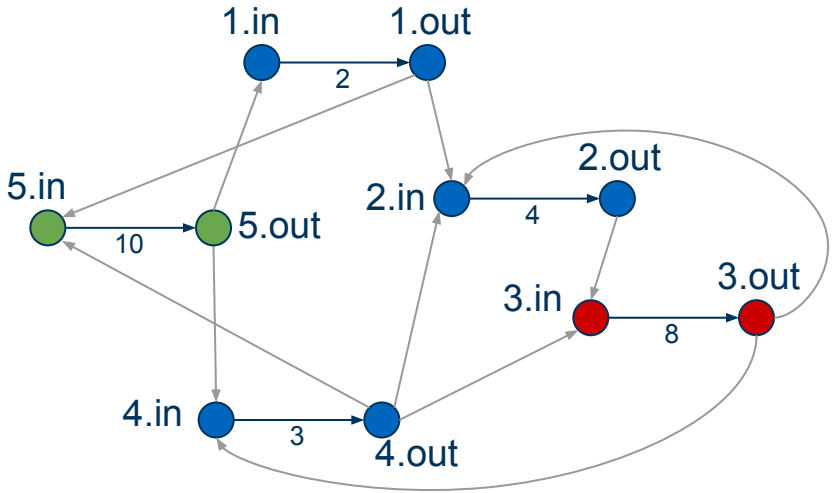


Now find Max-Flow

Graphs – Ford-Fulkerson

The idea is that any path from s to t in the residual network is an *augmenting path*.

Given such a path we can push flow equal to minimum of capacities on the path and increase the overall flow.

The bound on any implementation of this method on $\mathbb{Z}$ capacity values is $\mathcal{O}(m \cdot |f|)$ – finding a path is $\mathcal{O}(m)$ and each path increases the flow.

Graphs – Edmonds-Karp

Always choose the shortest path from s to t with a BFS.

$\mathcal{O}(nm^2)$.

Graphs – Dinitz

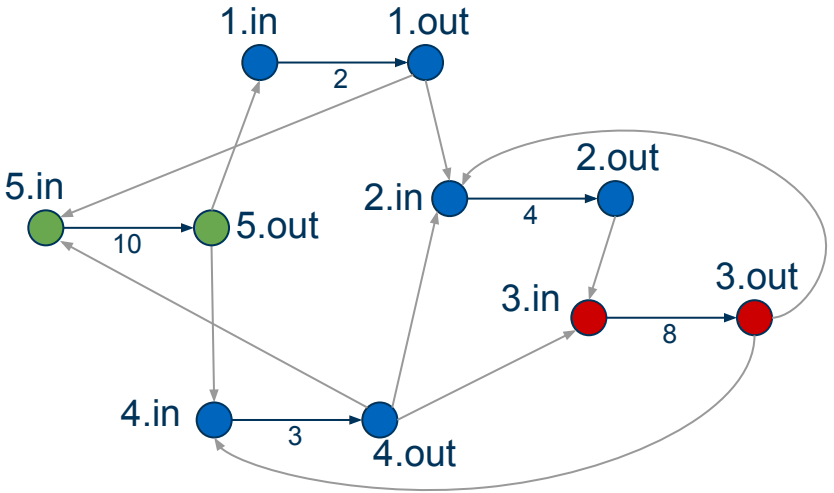
Also known as "Dinic" but Dinitz is correct.

Runs in $\mathcal{O}(n^2m)$ and even faster in special cases.

Idea – find a *blocking flow*, i.e. a *maximal* set of augmenting paths in the residual network.

More precisely:

- construct a DAG such that going forward always brings us closer to the sink;
- saturate the DAG with a series of DFS pushes.

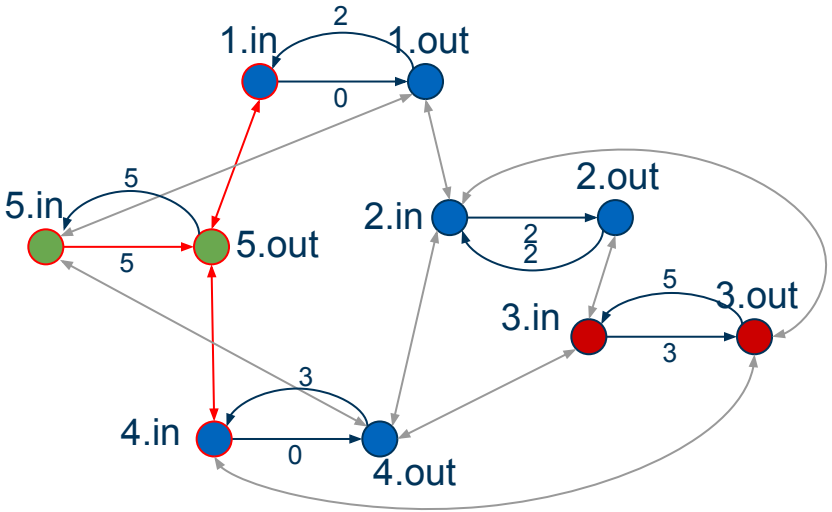


Find the Min-Cut from Max-Flow

1. **Run Dinic to completion** → you have a **max flow** and the final **residual graph**.

2. Do **one BFS/DFS from `s` in the residual graph**, following only edges with **residual capacity > 0**.
   - Let `S` = all vertices reachable from `s`.
   - Let `T` = `V \ S`.
3. Then `(S, T)` is a **minimum cut**.
   - The **cut edges** are all original edges `u -> v` with `u ∈ S` and `v ∈ T` whose **original capacity > 0** (equivalently: they will be **fully saturated** at max flow).

That's it. The extra work is **O(E)** (one graph traversal + one pass over edges).



Geometry

Geometry Primitives

Geometry problems reduce to **basic operations on points** in Euclidean space which maintain the foundations of analytical geometry.

Most algorithms are built from:

- vector arithmetic
- products (dot / cross)
- orientation tests

Points & Vectors

Points and Vectors in 2D

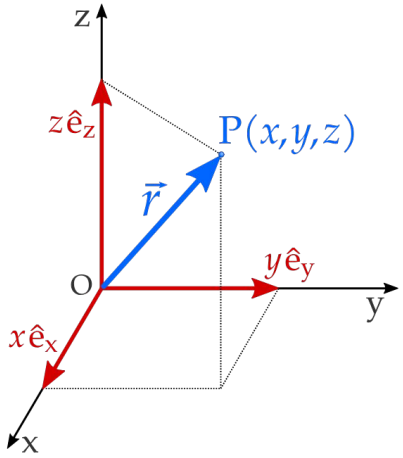
- A point is a pair of coordinates: $P=(x,y,z)$
- A vector is the difference of two points: $\vec{AB} = B - A = (x_B - x_A, y_B - y_A)$

Vector Operations:

- Addition / subtraction $\vec{u}+\vec{v}$, $\vec{u}-\vec{v}$
- Scalar multiplication $k\cdot\vec{u}$

In CP:

- Points and vectors are usually implemented as the same structure
- Operations are just integer arithmetic



Distance Between Points

Euclidean Distance

- For points $A(x_1,y_1)$, $B(x_2,y_2)$

$$d(A,B) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

- True geometric distance
- Needed only when the actual length is required

Squared Distance

$$d^2(A,B) = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

- Avoids square roots
- Preserves comparisons and ordering: $d(A,B) < d(C,D) \Leftrightarrow d^2(A,B) < d^2(C,D)$
- Faster and numerically safer than calculating sqrt.

Dot Product

Definition (Computation)

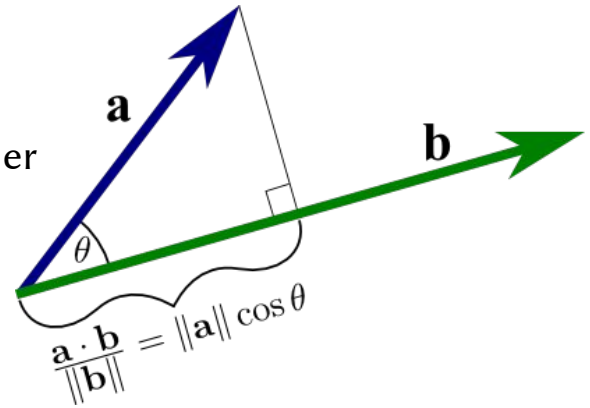
- For vectors $\vec{u}=(x_1,y_1),\vec{v}=(x_2,y_2)$:

$$\vec{u} \cdot \vec{v} = x_1x_2 + y_1y_2$$

Geometric Meaning

- Measures how much one vector points in the direction of another

$$\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos\theta$$



What It's Used For (CP)

- Angle classification:
  - acute / right / obtuse
- Projection of a point onto a line
- Distance from point to line

Cross Product & Orientations

While the dot product tells us about alignment, the cross product tells us about turning and orientation.

Cross Product (2D)

For vectors $\vec{u}=(x_1,y_1),\vec{v}=(x_2,y_2)$

$$\vec{u}\times\vec{v}=x_1y_2-y_1x_2$$

- Result is a scalar in 2D
- Sign and magnitude both carry meaning

Geometric Meaning

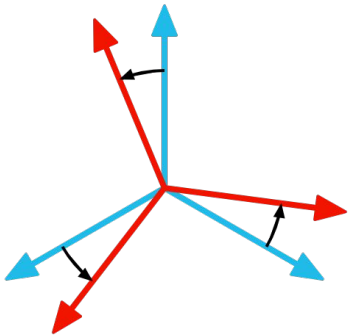
Signed Area: $|\vec{u}\times\vec{v}|=2\times$Area of the triangle formed by $\vec{u},\vec{v}$

- Larger magnitude $\longrightarrow$ larger area
- Zero $\longrightarrow$ vectors are collinear

Direction (Orientation)

The sign of the cross product tells us the turn direction:

- $\vec{u} \times \vec{v} > 0 \longrightarrow$ left turn
- $\vec{u} \times \vec{v} < 0 \longrightarrow$ right turn
- $\vec{u} \times \vec{v} = 0 \longrightarrow$ collinear



Orientation of Three Points

- Given three points A, B, C: Compute $(B - A) \times (C - A)$

Interpretation:

- $0 \longrightarrow A \longrightarrow B \longrightarrow C$ is a left turn
- $< 0 \longrightarrow$ right turn
- $= 0 \longrightarrow$ points are collinear

This is the orientation test.

From Triangles to Polygons

A polygon is a sequence of points $p_0, p_1, \ldots, p_{n-1}$ connected by line segments

- Edges: $(p_i, p_{i+1}), (p_{n-1}, p_0)$

Types

- Simple: no self-intersections
- Convex: all internal angles < 180°
- Non-convex: at least one reflex angle

Key idea:

- Polygons are built from edges and orientations

Oriented Area of a Triangle

Signed Area of Three Points

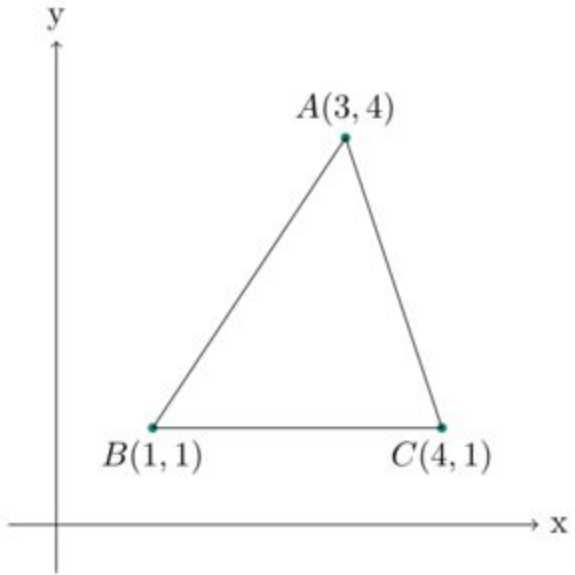
- Given points p_1, p_2, p_3: $2S = (p_2 - p_1) \times (p_3 - p_1))$

Interpretation:

- $S > 0 \longrightarrow$ counter-clockwise
- $S < 0 \longrightarrow$ clockwise
- $S = 0 \longrightarrow$ collinear

This formula gives:

- Orientation
- Triangle area
- This is the core primitive behind polygon algorithms



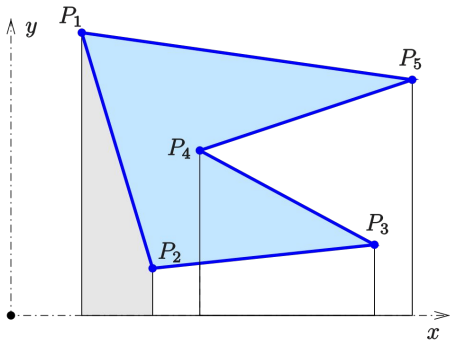
Polygon Area (Shoelace Formula)

Area of a Simple Polygon

- For vertices ordered consistently (CW or CCW):

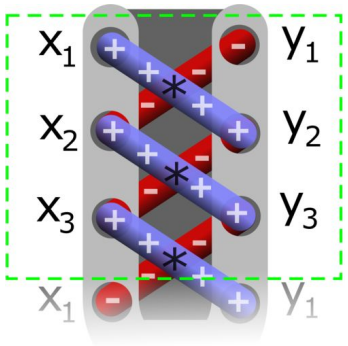
$$2A = \sum_{i=0}^{n-1} (x_i y_{i+1} - y_i x_{i+1})$$

(indices modulo n)



This is equivalent to:

- Summing signed triangle areas
- Using cross products of consecutive vertices



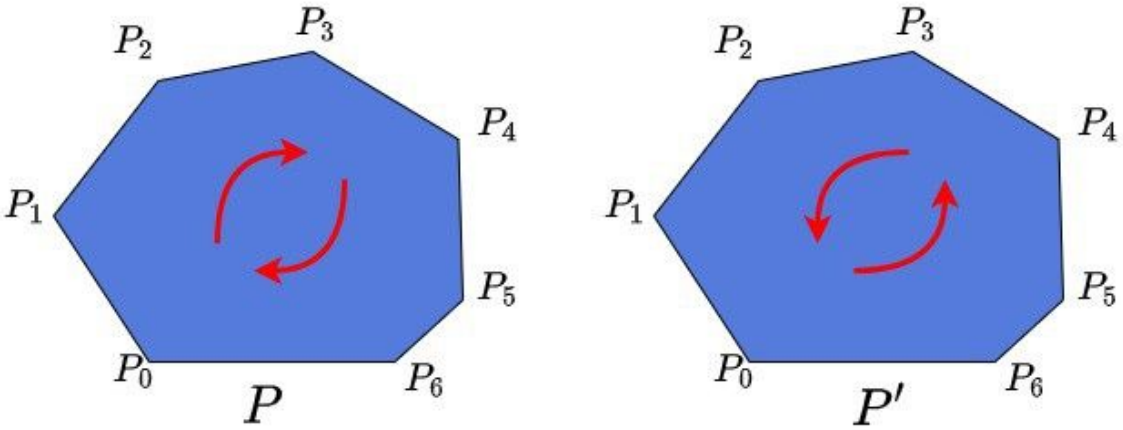
Properties:

- Works for any simple polygon
- Sign gives orientation
- $|A|$ is the actual area

Orientation of a Polygon

Clockwise vs Counter-Clockwise

- A polygon has global orientation
- Determined by the sign of its area $A > 0 \Longrightarrow$ CCW, $A < 0 \Longrightarrow$ CW



Convex vs Non-Convex Polygons

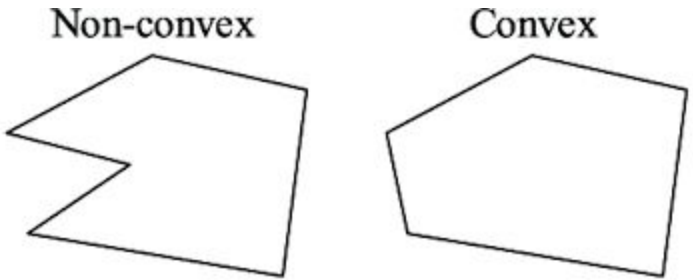
Convex Polygon

- Every turn is in the same direction
- For all triples: $(p_{i+1}-p_i)\times(p_{i+2}-p_{i+1})\geq 0$

- Binary search friendly

Non-Convex Polygon

- At least one “wrong” turn
- Still simple, but harder to process
- This distinction motivates convex hulls



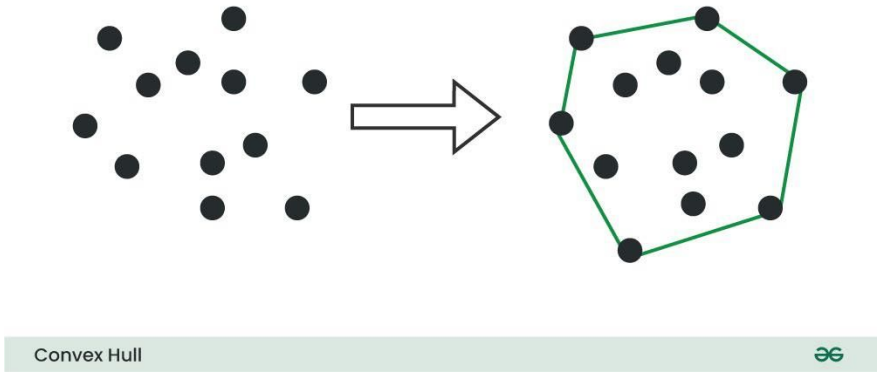
What Is a Convex Hull?

Given a set of points in the plane:

- The convex hull is the smallest convex polygon
- It contains all given points
- All interior points are irrelevant

Intuition:

- Stretching a rubber band around the points
- Points on the hull define the boundary



Convex Hull Construction: High-Level Idea

General Strategy

- Sort points in some order
- Traverse points one by one
- Maintain a candidate hull
- Remove points that break convexity

Key data structure:

- Stack / vector

Time complexity:

- Sorting: $O(n \log n)$
- Processing: $O(n)$

Convex Hull Construction: Graham Scan

Steps:

- Find the starting point $P0$:
  - Lowest y-coordinate
  - If tie: smallest x
- Sort all other points by:
  - Increasing polar angle around $P0$
  - Break ties by increasing distance from $P0$
- Traverse points in sorted order:
  - Maintain a stack of hull points
  - While the last three points do not form a clockwise turn:
    - Remove the middle point
- The stack contains the convex hull in clockwise order

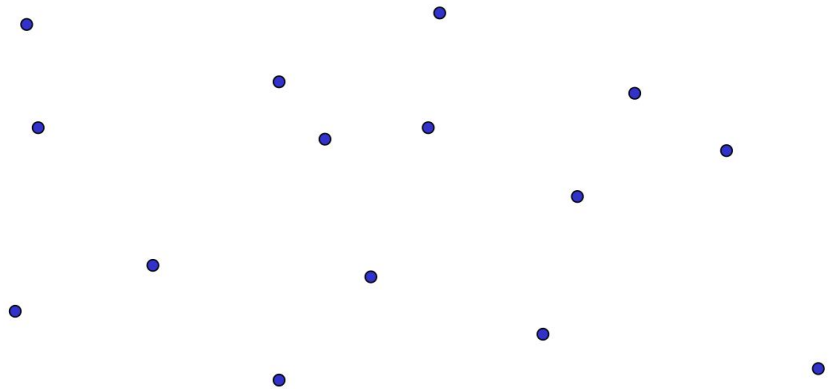
Key tool: **Orientation test** (cross product)

- Time complexity: $O(n \log n)$

Geometric divide-and-conquer

Closest Pair Problem:

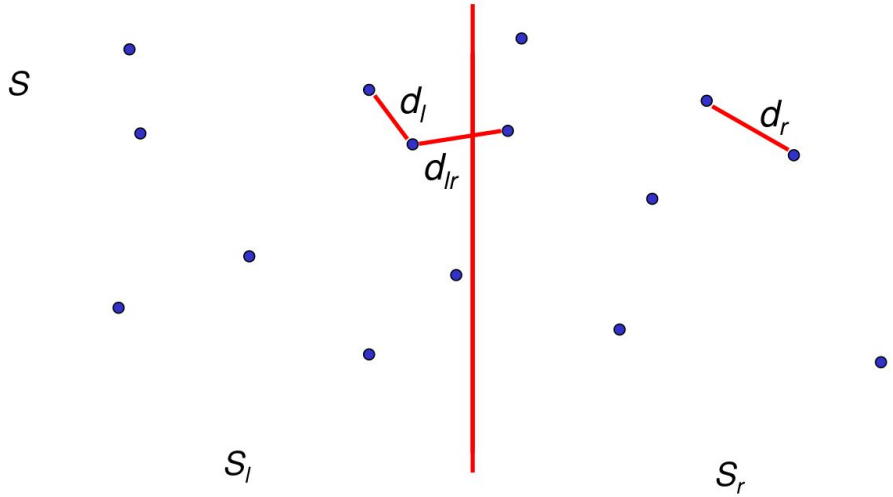
- Given a set S of n points in the plane, find a pair of points with the smallest distance.



Geometric divide-and-conquer

Divide and Conquer Method:

1. **Divide**: Divide S into two equal sized sets S_l und S_r .
2. **Conquer**: d_l = mindist(S_l), d_r = mindist(S_r)
3. **Merge**: d_{lr} = min{ d(p_l ,p_r) | $p_l \in S_l$, $p_r \in S_r$ }
4. return min{d_l , d_r , d_{lr} }



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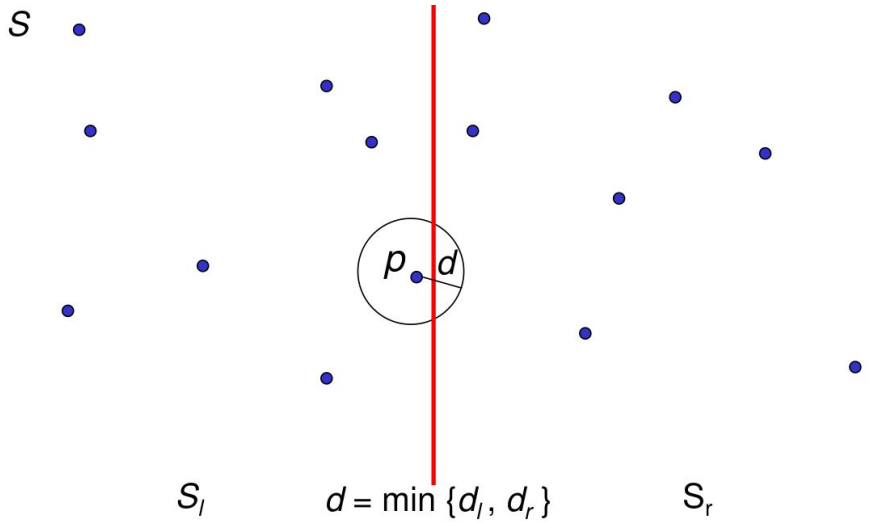
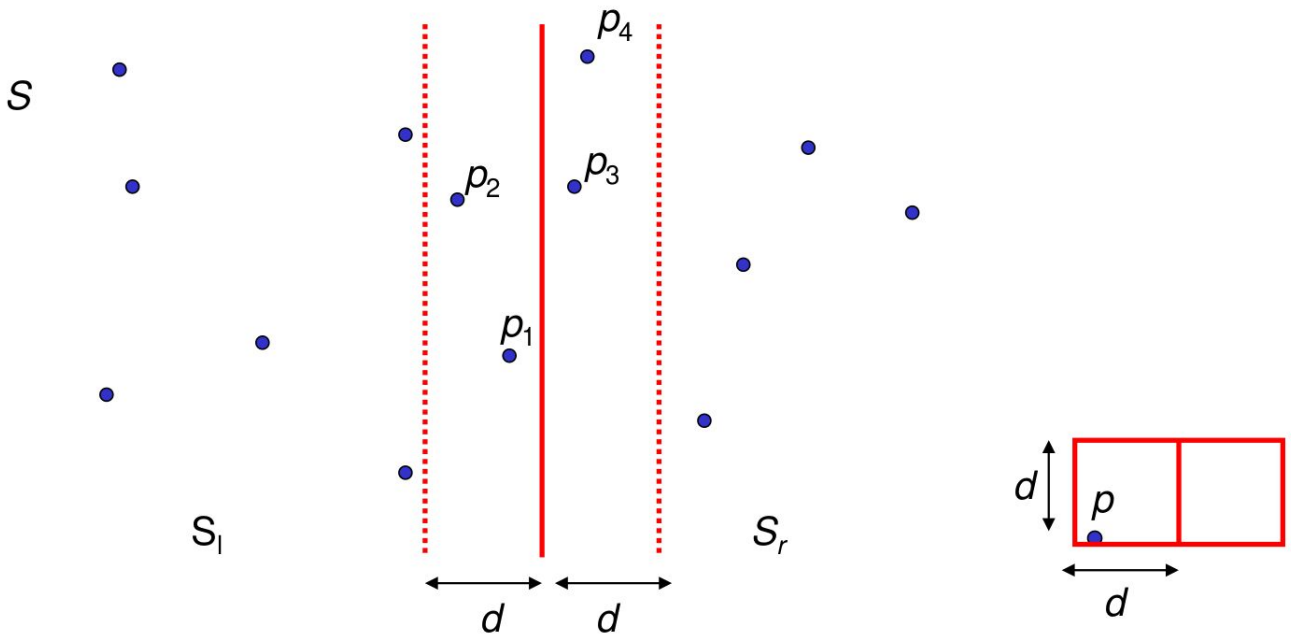


Merge Step



$d = \min \{ d_l, d_r \}$

Merge Step

- Initially sort the points in S in order of increasing x-coordinates O(n log n).
- Each bisection line can be determined in O(1) time.
- Once the subproblems S_l , S_r are solved, generate a list of the points in S in order of increasing y-coordinates. This can be done by **merging the sorted lists of points** of S_l , S_r (merge sort).

Segment Intersection: The Core Problem

Problem Statement

Given two segments:

- AB
- CD

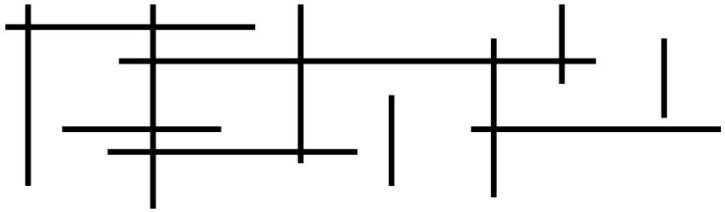
Determine whether they:

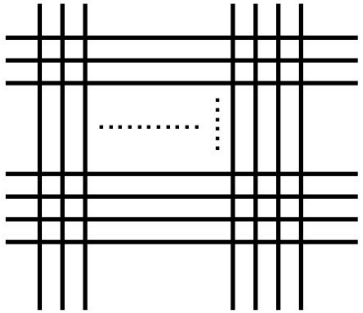
- intersect at a point
- overlap
- do not intersect

This is one of the most common geometry subproblems.

Line Segment Intersection

Find all pairs of intersecting line segments





Line Segment Intersection

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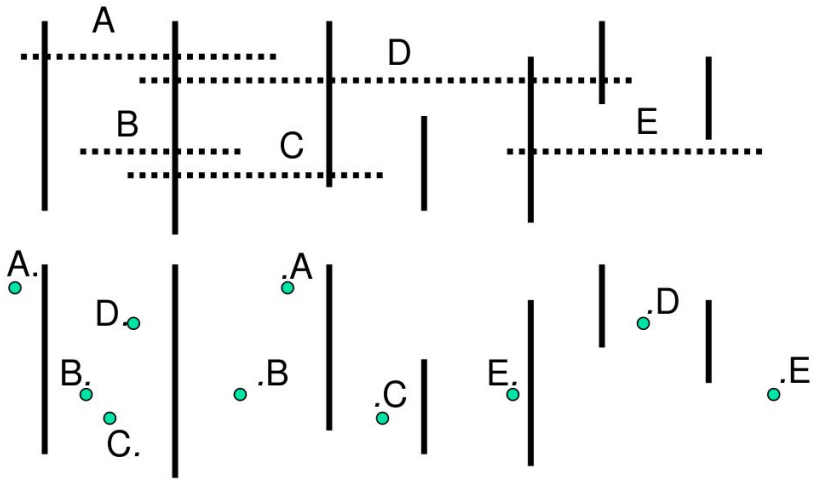


The representation of the horizontal line segments by their endpoints allows for a vertical partitioning of all objects.

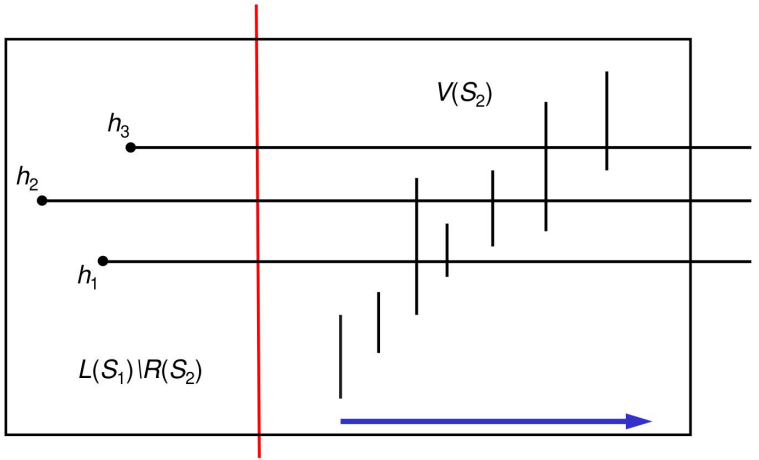
Output of intersections

Can be similarly solved with DnC

- Divide plane by a vertical line
- Solve intersections on left and right
- Detect cross-boundary intersections

Complexity:

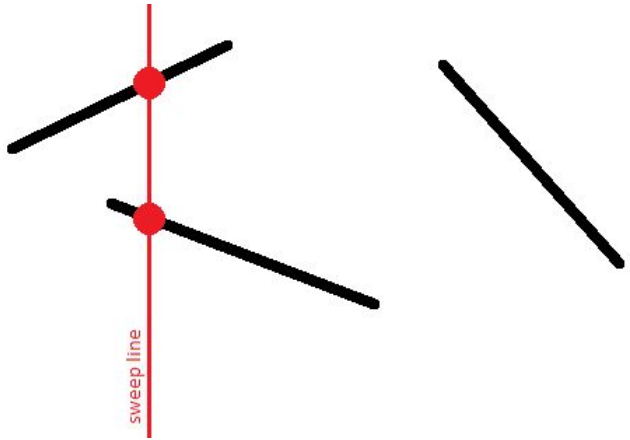
- $O(n\log n+k)$
- k = number of intersections



Another Approach: Sweep Line Algorithm

Idea: Let's draw a vertical line and start moving this line to the right.

- A segment becomes **active** when the sweep line reaches its left endpoint
- The segment remains active while the line intersects it
- It becomes inactive after passing its right endpoint



Assumptions:

- No vertical segments
- Each active segment intersects the sweep line at exactly one point

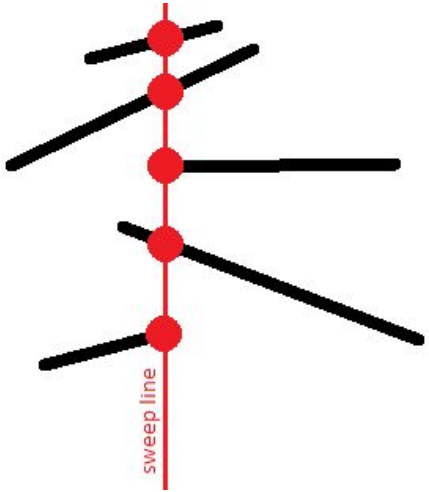
At any time, the sweep line maintains the set of currently active segments.

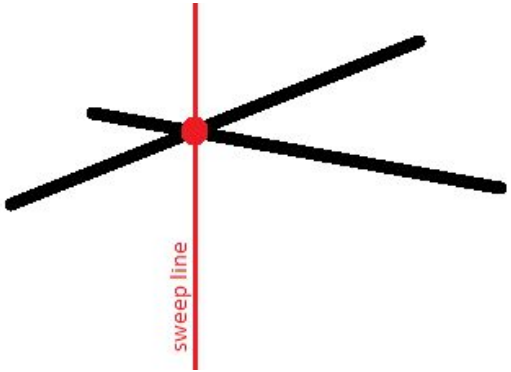
Sweep Line Algorithm

- At any time, consider all segments intersecting the sweep line
- Store these active segments ordered by their y-coordinate at the sweep line
- This vertical order changes only when segments intersect

Two segments intersect if and only if their **relative order** along the sweep line changes.

- Any intersecting pair of segments must become **neighbors** at some moment



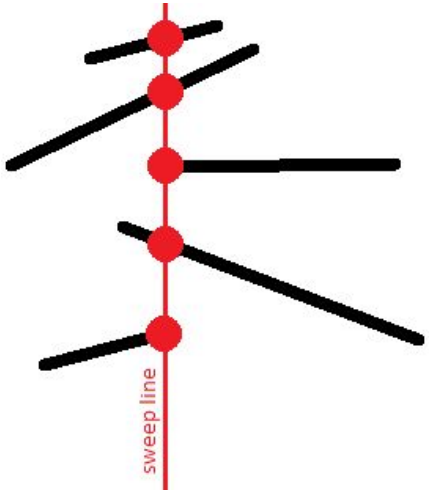


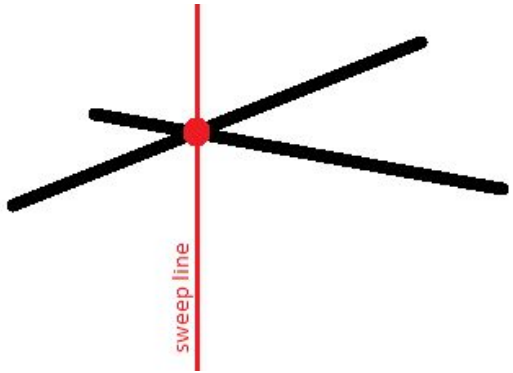
Sweep Line Algorithm: Concept

The set and order of active segments only change when:

- a segment starts
- a segment ends

Thus, it is sufficient to process the sweep line only at the x-coordinates of segment endpoints.





Handling Segment Insertion / Removal

When a segment starts:

- Insert it into the active set at the correct position (by y-order)
  Check for intersections with
  - its immediate neighbor above
  - its immediate neighbor below
- Only neighboring segments can intersect next
- No other segment's relative order is affected

When a segment ends:

- Remove it from the active set
- Its former upper and lower neighbors become adjacent
- Check these two segments for intersection

Checking only neighboring segments at events is sufficient.

Sweep Line for Segment Intersection

- Each segment:
  - is inserted once
  - is removed once

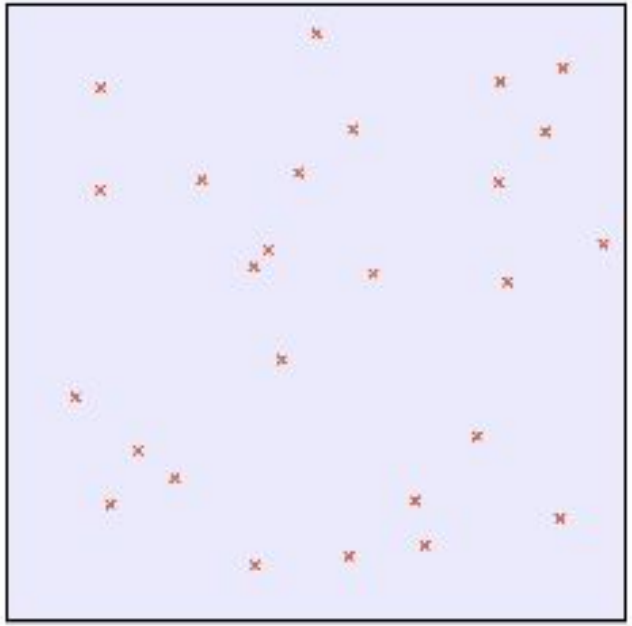
- Each event triggers a constant number of intersection checks
- Operations on the ordered structure take O(logn)

Final complexity: **O(nlogn)**

Sweep Line Applications

Sweep line is used for:

- Segment intersection
- Area of union of rectangles
- Closest pair of points
- Interval overlap in time
- Coverage problems



Assignment 5

Deadline: 14/01/2026

Mat vs. the Backyard Cat Gang

- Scare toys placed as points in the plane
- A point is protected if every direction moves closer to some toy
- Protected region depends on geometry of points

Constraints:

- n, m up to 100,000
- Need fast geometric updates

Happy Christmas !